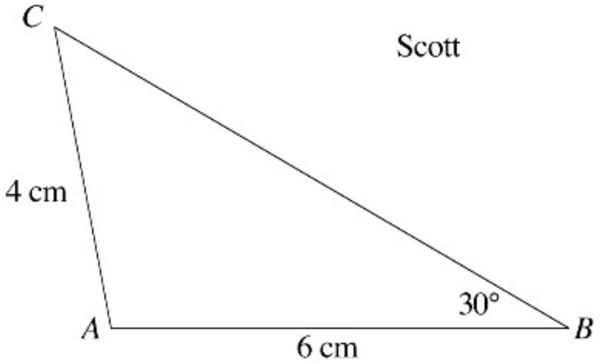
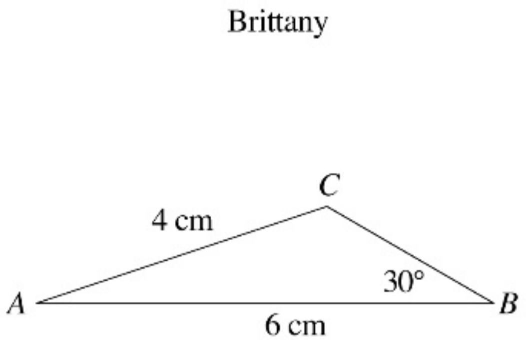
Trigonometry- Sine and Cosine Laws Lesson #4:
Problem Solving and The Ambiguous Case of The Sine Law

Introduction

Students were asked to determine the measure of angle ACB in a triangle in which $AB = 6$ cm, $AC = 4$ cm, and $\angle ABC = 30°$.

Two students, Scott and Brittany, each constructed a triangle to represent the given information.





a) Use a ruler and protractor to confirm that each student's triangle correctly represents the given information.

b) Without using a measuring device, estimate, to the nearest 10°, the measure of angle ACB in each case.

c) The work for Scott's diagram and for Brittany's diagram is identical in the first four steps as shown below:

$$\frac{\sin C}{c} = \frac{\sin B}{b} \quad \rightarrow \quad \frac{\sin C}{6} = \frac{\sin 30°}{4} \quad \rightarrow \quad \sin C = \frac{6 \sin 30°}{4} \quad \rightarrow \quad \sin C = 0.75$$

i) In Scott's case, use inverse sine to determine, to the nearest degree, the acute angle $\angle ACB$ whose sine ratio is 0.75.

acute $\angle ACB =$ ______°

ii) In Brittany's case, use inverse sine to determine, to the nearest degree, the obtuse angle $\angle ACB$ whose sine ratio is 0.75.

obtuse $\angle ACB =$ ______°

d) Calculate the measure of $\angle CAB$ in each case.

In this particular example, the information given allowed us to draw two different triangles which produced two different answers to the same question.

unclear about how to draw & label the triangle.

This is an example of **the ambiguous case of the sine law**.

But how do we know if given information will produce a unique solution or two different solutions? The investigations that follow will help us determine the answer to this question.

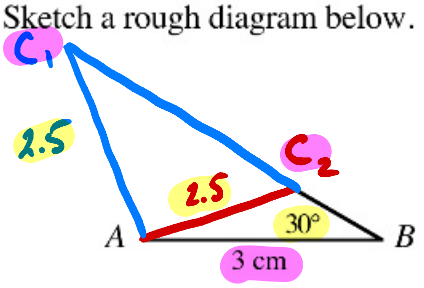
Investigating the Ambiguous Case of the Sine Law

In the previous investigations, we discovered that in the case of SSS, SAS, and ASA only one solution was possible. But in the case of SSA, zero, one, or two solutions are possible. We shall investigate the SSA scenario in this investigation.

Consider ΔABC in which $AB = 6$ cm and $\angle ABC = 30°$. We are going to consider three different measurements for AC and use the sine law to determine the measure of $\angle BAC$.

Case 1: $AC = 2.5$ cm

Step 1: Sketch a rough diagram below.



Step 2: Use the sine law to determine sin C.

$$\frac{\sin C}{c} = \frac{\sin B}{b}$$

$$\frac{\sin 30°}{2.5} = \frac{\sin C}{3}$$

$$\frac{3\sin 30°}{2.5} = \frac{2.5\sin C}{2.5}$$

Step 3: Solve $\sin C = 0.6$. +ive I, II

Reference angle = 37° or 180° − 37°

$\angle C = 37°$ or $\angle C = 143°$

$\angle ACB = 37°$ or $\angle ACB = 143°$

Step 4: Determine the measure of $\angle BAC$.

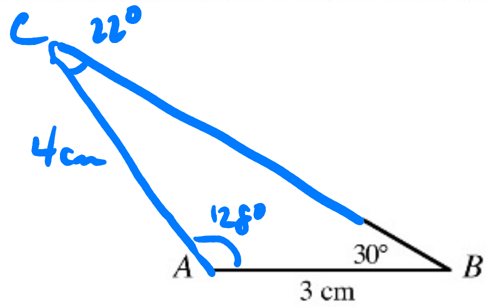
$\angle A = 180° - 30° - 143° = 7°$

$\angle A = 180° - 30° - 37° = 113°$

Step 5: State the solution to the problem.

Case 2: $AC = 4$ cm

Step 1: Sketch a rough diagram below.



Step 2: Use the sine law to determine sin C.

$$\frac{\sin C}{c} = \frac{\sin B}{b}$$

$$\frac{\sin 30°}{4} = \frac{\sin C}{3}$$

$$\frac{3\sin 30°}{4} = \frac{4\sin C}{4}$$

Step 3: Solve $\sin C = 0.375$ I, II

Reference angle = 22° ~~180° − 22° = 158°~~ too big

$\angle C = 22°$

$\angle ACB = 22°$

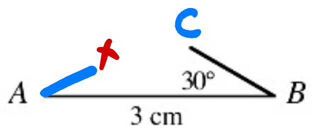
Step 4: Determine the measure of $\angle BAC$.

$\angle A = 180° - 30° - 22°$
$= 128°$

Step 5: State the solution to the problem.

Case 3: $AC = 1$ cm

Step 1: Sketch a rough diagram below.



Step 2: Use the sine law to determine sin C.

$$\frac{\sin C}{c} = \frac{\sin B}{b}$$

$$\frac{\sin C}{3} = \frac{\sin 30^\circ}{1}$$

$$\sin C = 3\sin 30^\circ$$

Step 3: Solve $\sin C =$ 1.5

$\sin^{-1}(1.5) = \emptyset$

Step 4: State the solution to the problem.

no triangles

* ① No diagram
* ② SSA
* ③ Ref ∠ > given ∠

Conditions for the Ambiguous Case of the Sine Law

The determining factor depends on how the reference angle compares to the given angle. Complete the following statements.

37° > 30° **i)** If the reference angle is greater than the given angle, there will be __2__ solution(s).

22° < 30° **ii)** If the reference angle is less than the given angle, there will be __1__ solution(s).

iii) If the reference angle does not exist, there will be __0__ solution(s).

Note

The case for two solutions can also be determined by looking at the two given sides.

Since $\frac{\sin C}{c} = \frac{\sin B}{b}$, then $\frac{b}{c} = \frac{\sin B}{\sin C}$.

In the case of b) above, where the reference angle C is less than the given angle B, the ratio $\frac{\sin B}{\sin C} > 1$, hence $\frac{b}{c} > 1$ and $b > c$.

This shows that:

- If the side opposite the given angle is greater than the side opposite the required angle, there is only one solution to the problem.
- If the side opposite the given angle is less than the side opposite the required angle, there are either two solutions or no solutions (depending on the height of the triangle).

Class Ex. #1

Without using the sine law, determine in which of the following cases there is **exactly one** solution to the problem, i.e. there is only one triangle which can be constructed from the given information.

a) Calculate $\angle C$ in ΔABC where $\angle B = 56°$, $c = 10$ cm, and $b = 12$ cm

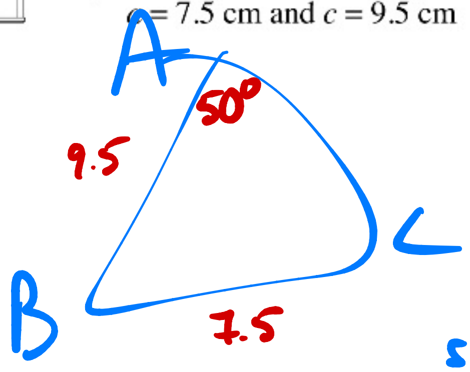
b) Calculate $\angle PQR$ in ΔPQR where $\angle QPR = 41°$, $RQ = 3$ cm, and $PR = 4.2$ cm

Class Ex. #2

Find all possible measures of $\angle C$ in the following triangles.

a) ΔABC where $\angle A = 50°$, $a = 7.5$ cm and $c = 9.5$ cm

b) ΔABC where $\angle A = 50°$, $a = 9.5$ cm and $c = 7.5$ cm



$$\frac{\sin 50°}{7.5} = \frac{\sin C}{9.5}$$

$$\frac{9.5\sin 50°}{7.5} = \frac{7.5\sin C}{7.5}$$

$\sin C = 0.9703$

ref∠ = 76° > 50°

∠C = 76°, 104° 2 triangles

∠B = 54°, 26°

Complete Assignment Questions #1 - #6

$\boxed{\#1\text{-}5 \text{ (omit \#2)}}$

Assignment

1. Explain what is meant by the ambiguous case of the sine law. Describe situations in which a sine law problem may have no solution, one solution or two solutions.