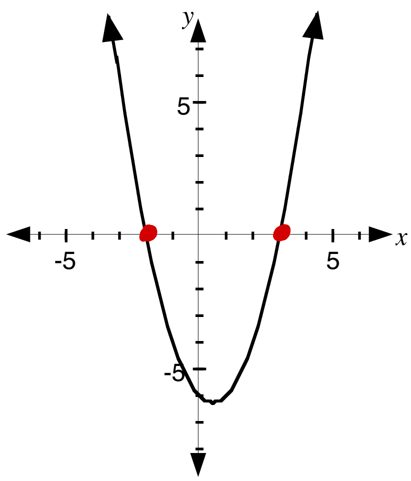
Polynomial Functions and Equations Lesson #5:
Factoring Polynomial Expressions - Part One

In this lesson we will focus on factoring polynomial expressions where the leading coefficient is 1.

Review

Consider the polynomial function $f(x) = x^2 - x - 6$. The function can be expressed in factored form as $f(x) = (x + 2)(x - 3)$.

a) Fill in the blanks in the illustration below.



The ***graph*** of	The ***function***	The ***equation***
$f(x)=x^2-x-6$	$f(x)=x^2-x-6$ $= (x+2)(x-3)$	$x^2-x-6=0$
has ***x-intercepts***	has ***zeros***	has ***roots***
$x =$ -2 and $x =$ 3	-2 and 3	$x =$ -2 and $x =$ 3

b) Fill in the blanks in the following statement regarding the function with equation $y = f(x)$.

" The _________ of the function, the ___________ of the graph of the function, and the __________ of the corresponding equation $y = 0$, are the ________ numbers."

Investigating the Integral Zero Theorem

a) The factors of the polynomial, $P(x) = x^2 + 5x - 24$ are $(x + 8)$ and $(x - 3)$.

i) State the zeros of the polynomial function. -8 & 3

ii) What do you notice about the constant term in the polynomial and the zeros of the polynomial?

-8 & 3 are factors of the constant (-24)

b) The factors of the polynomial, $P(x) = x^3 - 2x^2 - x + 2$ are $(x - 2)$, $(x - 1)$, and $(x + 1)$.

i) State the zeros of the polynomial function. 2, 1, -1

ii) What do you notice about the constant term in the polynomial and the zeros of the polynomial?

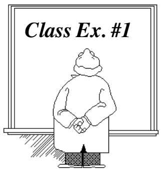
all factors of the constant (2)

The examples on the previous page are illustrations of the **Integral Zero Theorem** which states that **the zeros of an integral polynomial function with a leading coefficient of 1 must be factors of the constant term**.

Consider $P(x) = x^3 - 7x - 6$.

- The integral zero theorem tells us that zeros of $P(x)$ must be factors of -6 and so must come from the list $\pm 1, \pm 2, \pm 3, \pm 6$.
- This list contains all **potential zeros** of the polynomial.

Class Ex. #1



Consider the polynomial $P(x) = x^3 - 7x - 6$.

a) List the potential zeros of the polynomial. $\pm 1, \pm 2, \pm 3, \pm 6$

b) List the potential factors of the polynomial.

$x+1$ $\quad x+2$ $\quad x+3$ $\quad x+6$

$x-1$ $\quad x-2$ $\quad x-3$ $\quad x-6$

c) Students were asked to identify one of the actual factors of the polynomial.
Luke used the factor theorem, and Nicole used synthetic division. Their work is shown.

Luke's Work Using the Factor Theorem

- Try $x - 1$ as a factor.

$P(1) = (1)^3 - 7(1) - 6 = -12$

$P(1) \neq 0$, therefore $x - 1$ is not a factor.

- Try $x + 1$ as a factor.

$P(-1) = (-1)^3 - 7(-1) - 6 = 0$

$P(1) = 0$, therefore $x + 1$ is a factor.

- Use $x + 1$ as a factor in synthetic division to find a quadratic quotient

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -7 & -6 \\ & & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & 0 \end{array}$$

$(x+1)(x^2 - x - 6)$

Nicole's Work Using Synthetic Division

- Try $x - 1$ as a factor.

$$\begin{array}{r|rrrr} 1 & 1 & 0 & -7 & -6 \\ & & 1 & 1 & -6 \\ \hline & 1 & 1 & -6 & -12 \end{array}$$

Since the remainder is -12, then $x - 1$ is not a factor.

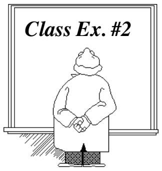
- Try $x + 1$ as a factor.

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -7 & -6 \\ & & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & 0 \end{array}$$

Since the remainder is 0, then $x + 1$ is a factor.

Complete the factoring of the polynomial $x^3 - 7x - 6$.

$\boxed{(x+1)(x-3)(x+2)}$



Class Ex. #2

Consider the polynomial $P(x) = x^4 + 2x^3 - 7x^2 - 8x + 12$.

a) List the potential zeros of the polynomial. $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$

b) Express the polynomial in factored form.

$P(1) = 0$

1	1	2	−7	−8	12
	↓	1	3	−4	−12
	1	3	−4	−12	$\boxed{0}$

$(x-1)(x^3+3x^2-4x-12)$

$P(2) = 0$

2	1	3	−4	−12
	↓	2	10	12
	1	5	6	$\boxed{0}$

$(x-1)(x-2)(x^2+5x+6)$

$(x-1)(x-2)(x+3)(x+2)$

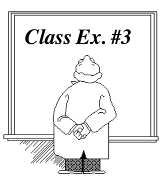
c) State the zeros of $P(x)$.

$1, 2, -3, -2$

d) State the roots of the equation $x^4 + 2x^3 - 7x^2 - 8x + 12 = 0$.

$x = 1, 2, -3, -2$

Complete Assignment Questions #1 - #8



Class Ex. #3

Consider the polynomial function $P(x) = x^3 - 3x^2 - 3x + 1$. $\rightarrow \pm 1$

a) Find the zeros of $P(x)$, to the nearest hundredth, using a graphing calculator.

b) Use the integral zero in a) and synthetic division to determine the exact value of the zeros.

$P(-1) = 0$

−1	1	−3	−3	1
	↓	−1	4	−1
	1	−4	1	$\boxed{0}$

$(x+1)(x^2-4x+1)$

↓

CNF

↓

quad formula

$x = -1.00$

$x = 3.73$

$x = 0.27$

Complete Assignment Questions #9 - #13

$\boxed{\#1\text{–}6, 9\text{–}12}$

Polynomial Functions and Equations Lesson #6:
Factoring Polynomial Expressions - Part Two

In this lesson we extend the factoring of polynomial expressions to examples where the leading coefficient is not 1.

Investigating the Rational Zero Theorem

$a \neq 1$

a) The factors of the polynomial $P(x) = 15x^2 - 34x + 16$ are $(3x - 2)$ and $(5x - 8)$.

i) State the zeros of the polynomial function. $2/3$ $8/5$

ii) What do you notice about the constant term in the polynomial and the constant terms in the factors?

2 & 8 are factors of the constant (16)

iii) What do you notice about the leading coefficient in the polynomial and the leading coefficients in the factors?

3 & 5 are factors of the leading coefficient (15)

iv) How do the zeros relate to the coefficients in the polynomial?

b) The factors of the polynomial $P(x) = 10x^3 + 81x^2 - 298x + 231$ are $(2x - 3)$, $(5x - 7)$, and $(x + 11)$.

i) State the zeros of the polynomial function.

ii) What do you notice about the constant term in the polynomial and the constant terms in the factors?

iii) What do you notice about the leading coefficient in the polynomial and the leading coefficients in the factors?

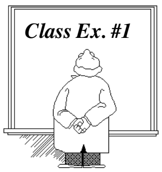
iv) How do the zeros relate to the coefficients in the polynomial?

The examples above are illustrations of the **Rational Zero Theorem** which states that **the zeros of an integral polynomial function $P(x)$ are of the form $\frac{p}{q}$, where the numerator p is a factor of the constant term, and the denominator q is a factor of the leading coefficient**.

i.e. $$\frac{p}{q} = \frac{\text{Factor of the constant term}}{\text{Factor of the leading coefficient}}$$

Consider the polynomial $P(x) = 5x^3 - 51x^2 + 55x - 9$.

- The factors of the constant term, -9, are: $\pm1, \pm3, \pm9$.
- The factors of the leading coefficient, 5, are: $\pm1, \pm5$.
- $\dfrac{p}{q} \Rightarrow \dfrac{\text{Factors of the constant term}}{\text{Factors of the leading coefficient}} \Rightarrow \dfrac{\pm1, \pm3, \pm9}{\pm1, \pm5}$
- **Potential zeros** of $P(x)$ are $\pm1, \pm3, \pm9, \pm\frac{1}{5}, \pm\frac{3}{5}, \pm\frac{9}{5}$.



Consider the function $P(x) = 5x^3 - 51x^2 + 55x - 9 = 0$.

a) Use the potential zeros above to list the potential binomial factors of $P(x)$ in the form $ax - b$, where $a \in N$ and $b \in I$.

b) Show that $x = 1$ is a zero of $P(x)$, and hence determine all the zeros of $P(x)$.

$P(1) = 0$

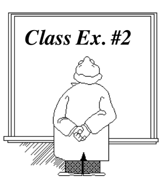
$$\begin{array}{c|cccc} 1 & 5 & -51 & 55 & -9 \\ & \downarrow & 5 & -46 & 9 \\ \hline & 5 & -46 & 9 & \boxed{0} \end{array}$$

$(x-1)(5x^2-46x+9)$

$(x-1)(5x-1)(x-9)$

c) Solve the equation $5x^3 - 51x^2 + 55x - 9 = 0$.

$\boxed{x = 1, \frac{1}{5}, 9}$



Determine all real solutions to the equation $2x^3 - x^2 - x - 3 = 0$.

$$\begin{array}{c|cccc} \frac{3}{2} & 2 & -1 & -1 & -3 \\ & \downarrow & 3 & 3 & 3 \\ \hline & 2 & 2 & 2 & \boxed{0} \end{array}$$

$\dfrac{p}{q} = \dfrac{\pm1, \pm3}{\pm1, \pm2}$

$= \pm1, \pm3, \pm\frac{1}{2}, \pm\frac{3}{2}$

$P\left(\frac{3}{2}\right) = 0$

$(x-\frac{3}{2})(2x^2+2x+2)$

$(x-\frac{3}{2})2(x^2+x+1)$

$\boxed{x = \frac{3}{2}}$

$(2x-3)(x^2+x+1) \rightarrow$ CNF $\rightarrow$ quad. formula $\rightarrow x = \varnothing$

Complete Assignment Questions #1 - #12