Polynomial Functions and Equations Lesson #4:
The Remainder Theorem and the Factor Theorem

Review

a) Use synthetic division to divide $P(x) = x^3 - 2x^2 - 4$ by $x + 1$.

Calculate $P(-1)$ in $P(x) = x^3 - 2x^2 - 4$.

b) Use synthetic division to divide $P(x) = x^2 - 2x - 5$ by $x - 2$.

Calculate $P(2)$ in $P(x) = x^2 - 2x - 5$.

c) Complete the following statements based on your observations in a) and b).

- When $P(x) = x^3 - 2x^2 - 4$ is divided by $x + 1$, the __________ is equal to $P(\quad)$.
- When $P(x) = x^2 - 2x - 5$ is divided by $x - 2$, the __________ is equal to $P(\quad)$.

The Remainder Theorem

P(x) = remainder.

When a polynomial function, $P(x)$, is divided by a binomial, $(x - a)$, the remainder obtained is equal to the value of the polynomial when $x = a$, i.e. the remainder is $P(a)$.

Proof:

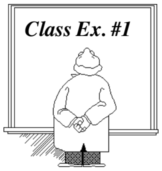
The division algorithm states $P(x) = D(x) \cdot Q(x) + R(x)$

Using $x - a$ as the divisor, we get $P(x) = (x - a) \cdot Q(x) + R(x)$

To find $P(a)$ we can substitute a for x to get

$$\begin{aligned} P(a) &= (a - a) \cdot Q(a) + R \\ &= 0 \cdot Q(a) + R \\ &= 0 + R \end{aligned}$$

$\therefore P(a) = R$ which is what the remainder theorem states.



Use the remainder theorem to find the remainder when $P(x) = 6x^3 - 4x^2 + 8x + 6$ is divided by

i) $x + 1$

$P(-1) = 6(-1)^3 - 4(-1)^2 + 8(-1) + 6$

$= -12$

$\downarrow$

remainder

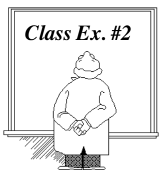
ii) $2x - 1$

$P\left(\frac{1}{2}\right) = 6\left(\frac{1}{2}\right)^3 - 4\left(\frac{1}{2}\right)^2 + 8\left(\frac{1}{2}\right) + 6$

$= \frac{39}{4}$

$\downarrow$

remainder



Find a if the remainder is 131 when $P(x) = 2x^4 - x^3 - ax + 8$ is divided by $x - 3$;

a) using synthetic division

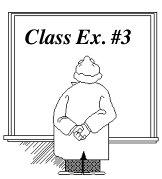
b) using the remainder theorem $\to P(3) = 131$

$131 = 2(3)^4 - (3)^3 - 3a + 8$

$3a = 162 - 27 + 8 - 131$

$\frac{3a}{3} = \frac{12}{3}$

$\boxed{a = 4}$



Find the coefficients d and c in $P(x) = 2x^4 + dx^3 - cx^2 + 5x - 8$ if the remainder is -41 when divided by $x + 3$ and the remainder is 74 when divided by $x - 2$.

Complete Assignment Questions #1 - #5

The Factor Theorem

The binomial $x - a$ is a **factor** of the polynomial function $P(x)$ if, and only if, $P(a) = 0$.
Note that a is then a **zero** of the polynomial function $P(x)$.

Class Ex. #4

Show that $x - 4$ is a factor of $P(x) = x^2 + 2x - 24$ by using

a) synthetic division

$$\begin{array}{c|ccc} 4 & 1 & 2 & -24 \\ & \downarrow & 4 & 24 \\ \hline & 1 & 6 & \boxed{0} \end{array}$$

$x + 6$

b) the factor theorem

$P(4) = 4^2 + 2(4) - 24$

$= \boxed{0}$

$\therefore$ $x - 4$ is a factor

Class Ex. #5

Write a binomial factor with integral coefficients of the polynomial $P(x)$ if

a) $P(3) = 0$

$x - 3$ is a factor

b) $P\left(-\frac{2}{3}\right) = 0$ $\longrightarrow 3 \cdot \left(x + \frac{2}{3}\right) = 3x + 2$

Class Ex. #6

If $P(5) = P(-2) = 0$, determine a second degree factor of the polynomial $P(x)$.

$(x - 5)(x + 2) = \boxed{x^2 - 3x - 10}$

Class Ex. #7

Use the factor theorem to determine which of the following is a factor of $4x^3 - 16x^2 - x + 4$.

a) $x + 2$

$P(-2) = -90$

$\therefore$ $x + 2$ is not a factor.

b) $2x - 1$

$P\left(\frac{1}{2}\right) = 0$

$\therefore$ $2x - 1$ is a factor.

Class Ex. #8

Show that 1 is a root of the equation $x^3 - 9x^2 + 20x - 12 = 0$ and find the other roots.

$P(1) = 0$

$\therefore$ $x - 1$ is a factor

$$\begin{array}{c|cccc} 1 & 1 & -9 & 20 & -12 \\ & \downarrow & 1 & -8 & 12 \\ \hline & 1 & -8 & 12 & \boxed{0} \end{array}$$

$\underbrace{1 \quad -8 \quad 12}_{x^2 - 8x + 12}$

$(x - 1)(x^2 - 8x + 12)$

$\boxed{(x - 1)(x - 6)(x - 2)}$

#1-4, 7, 8, 10, 11