Analyzing Radical and Rational Functions Lesson #3: Rational Functions - Asymptotes

Rational Functions

A rational function is a function which takes the form $f(x) = \dfrac{n(x)}{d(x)}$ where $n(x)$ and $d(x)$ are polynomial functions and $d(x) \neq 0$. The degree of $d(x)$ needs to be greater than zero or the function $f(x)$ is simply a polynomial function.

Examples of rational functions are $f(x) = \dfrac{x+2}{x-1}$, $g(x) = \dfrac{1}{x^2-9}$, $h(x) = \dfrac{x^2-4}{x+2}$.

Discontinuities

Since rational functions are expressed as fractions, the denominator cannot equal zero.

restrictions on the domain

If the denominator of a rational function can be made equal to zero for a particular value of the independent variable, then this value of the variable is called a **non-permissible value** of the function.
In the examples above, the non-permissible values are 1, ±3, and –2, respectively.

Note that not all rational functions have non-permissible values for the independent variable, e.g. $f(x) = \dfrac{1}{x^2+1}$.

If a rational function has non-permissible values, then the graph of the rational function will not be a **continuous** curve. There must be some kind of **discontinuity** in the graph.

In this course we will learn about two types of discontinuity:

i) **Infinite Discontinuity** (leading to a **vertical asymptote** on the graph)

ii) **Point Discontinuity** (leading to a **hole** on the graph)

In this lesson, we will examine the behaviour of the graphs of rational functions which have infinite discontinuities.

In the next lesson, we will examine the behaviour of the graphs of rational functions which have point discontinuities.

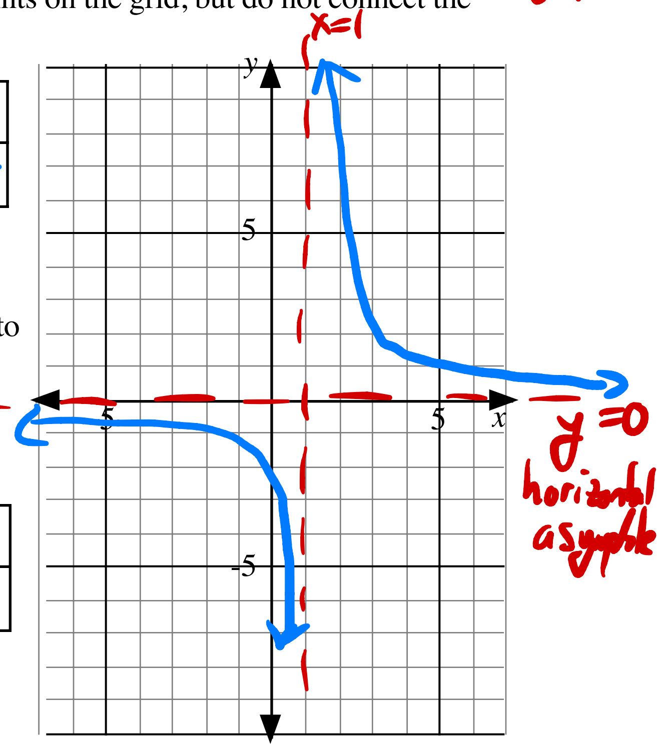
Graphing a Rational Function using a Table of Values

Consider the rational function $f(x) = \dfrac{1}{x-1}$.

a) Since division by zero is not defined, the non-permissible value of the function is 1 . vertical asymptote

b) Complete the table of values below. Plot the points on the grid, but do not connect the points at this time.

x	-5	-2	-1	0	1	2	3	4	5
$f(x)$	$-\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	∅	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$



c) To investigate the behaviour of the rational function near the non-permissible value, we include values of the independent variable close to the non-permissible value.

Complete the table below, plot the points on the grid, and join the points with a smooth curve.

x	0.5	0.9	0.99	1	1.01	1.1	1.5
$f(x)$	-2	-10	-100	∅	100	10	2

d) Verify the graph using a graphing calculator.

e) Describe what happens to the value of the function as x gets more and more positive, i.e. as $x \to \infty$ (read "as x approaches infinity").

the function approaches zero

f) Describe what happens to the value of the function as x gets more and more negative i.e. as $x \to -\infty$.

" "

g) From the options in the curly brackets, circle the correct alternative in each of the statements below.

- As x gets closer and closer to the non-permissible value of 1, from the left, the value of the function approaches { ∞ / $-\infty$ }.

- As x gets closer and closer to the non-permissible value of 1, from the right, the value of the function approaches { ∞ / $-\infty$ }.

Asymptotes

A line that a curve approaches more and more closely is called an **asymptote**.

With reference to the exploration on the previous page.

- As x gets closer and closer to 1, the graph of $f(x)$ gets closer and closer to the line with equation $x = 1$ but will never reach the line $x = 1$.

  The graph has an infinite discontinuity at $x = 1$ and the line $x = 1$ is called a **vertical asymptote** of the graph of $f(x)$.

  This occurs because 1 is a zero of the denominator of the rational function.

- As $|x|$ increases in value (i.e. as $x \to \pm\infty$), the graph of $f(x)$ gets closer and closer to the x-axis (the line with equation $y = 0$) but will never reach the x-axis.

  The line $y = 0$ is a **horizontal asymptote** of the graph of $f(x)$.

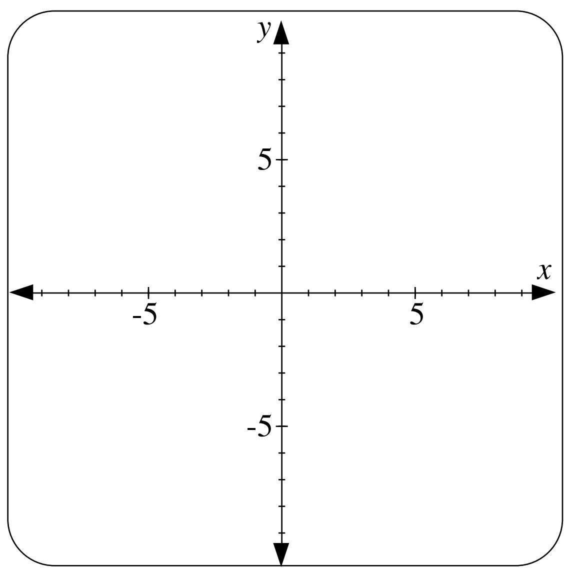
  Note that a horizontal asymptote is not representative of an infinite discontinuity. can be crossed.

- Vertical and horizontal asymptotes are often represented on graphs with dashed lines, but these dashed lines do not form part of the graph.

Investigating Asymptotes on the Graphs of Rational Functions

Part 1

Consider the function $f(x) = \dfrac{3}{x}$.



a) State the domain of the function.

b) Use a graphing calculator to graph the function, but <u>do not sketch</u> on the grid.

c) Write the equation of the vertical asymptote using the information from a) and b).

d) Show the vertical asymptote on the grid with a dashed line.

e) Complete the graph on the grid.

f) State the equation of the horizontal asymptote.

g) State the range of the function.

Algebracially Determining the Equations of Asymptotes

Investigations Part 1 to Part 3 and Assignment Questions #1 - #3 are examples of the following rules which can be used to algebraically determine the equations of vertical and horizontal asymptotes of rational functions.

These rules apply for rational functions of the form $f(x) = \dfrac{n(x)}{d(x)}$ **provided that $n(x)$ and $d(x)$ have no factors in common**. Situations where the numerator and denominator have a factor in common will be dealt with under **point discontinuity** in the next lesson.

Vertical Asymptotes

Algebraically we can find the equations of vertical asymptotes of rational functions by finding the zeros of the denominator because the graph is undefined at those value(s). The equation(s) will be x = *the zero value(s)* of the denominator.

all restricted values

Horizontal Asymptotes

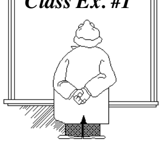
The graph of $f(x)$ has a horizontal asymptote under the following conditions:

* memorize

- If the degree of $n(x)$ is less than the degree of $d(x)$, then the line $y = 0$ is a horizontal asymptote. See Investigation Part 1 where $f(x) = \dfrac{3}{x}$.
- If the degree of $n(x)$ is equal to the degree of $d(x)$, then the line $y = \dfrac{a}{b}$ is a horizontal asymptote, where a is the leading coefficient of $n(x)$ and b is the leading coefficient of $d(x)$. See Investigation Part 2 where $f(x) = \dfrac{2x}{x+1}$.

  $a = 2$
  $b = 1$
  asymptote at $y = 2$
- If the degree of $n(x)$ is greater than the degree of $d(x)$, then the graph has no horizontal asymptote. See Investigation Part 3 where $f(x) = \dfrac{x^2}{x-2}$.

Class Ex. #1



Algebraically determine the equations of the asymptotes of the graph of the function $f(x) = \dfrac{x^2 + x - 6}{2x^2 - x - 3}$. Verify using a graphing calculator.

FRS

$f(x) = \dfrac{(x+3)(x-2)}{(2x-3)(x+1)}$

$x \neq \frac{3}{2}, -1$

hor. asymptote @ $y = \frac{1}{2}$

vert. asymptotes @ $x = \frac{3}{2}, -1$

Complete Assignment Questions #4 - #9

$\boxed{\#1, 2, 4, 6}$

Analyzing Radical and Rational Functions Lesson #4: Rational Functions - Points of Discontinuity

Review

Recall the following from the previous lesson.

If the denominator of a rational function can be made equal to zero for a particular value of the independent variable, then this value of the variable is called a **non-permissible value**.

If a rational function has non-permissible values, then the graph of the rational function will not be a **continuous** curve. There must be some kind of **discontinuity** in the graph.

Investigating Points of Discontinuity

Part 1: Consider the function $f(x) = \dfrac{x^2 - 4x + 2}{(x - 3)}$ and $g(x) = \dfrac{x^2 - 4x + 3}{(x - 3)}$.

a) State the non-permissible value of each function. $x \neq 3$

b) Use a graphing calculator to graph $y = f(x)$ and sketch the graph on Grid 1.

c) State the equation of the asymptote of the graph of f. $x = 3$

d) A student assumed that the graph of $y = g(x)$ would be similar to the graph of $y = f(x)$ with a vertical asymptote with equation $x = 3$. To investigate the student's assumption, graph $y = g(x)$ using a graphing calculator, and answer the following questions.

- What shape does the graph of $y = g(x)$ appear to take? linear
- Does the graph of $y = g(x)$ have a vertical asymptote? no
- Explain why the graph of $y = g(x)$ must have a discontinuity.

$$g(x) = \frac{\cancel{(x-3)}(x-1)}{\cancel{x-3}}$$

e) The discontinuity on the graph can be seen by using the use zoom decimal window feature of the graphing calculator (press $\boxed{\text{ZOOM}}$ $\boxed{4}$).

$g(x) = x - 1$

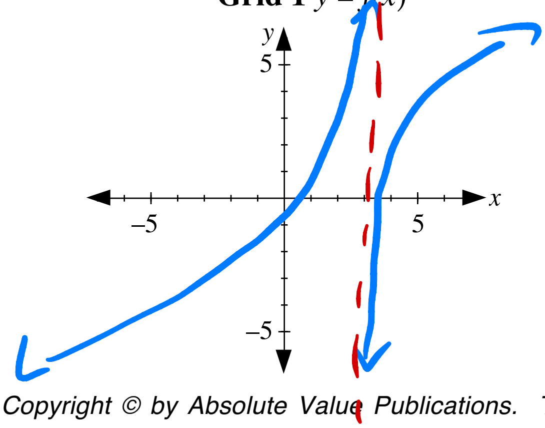
$x = 3$

$y = 2$

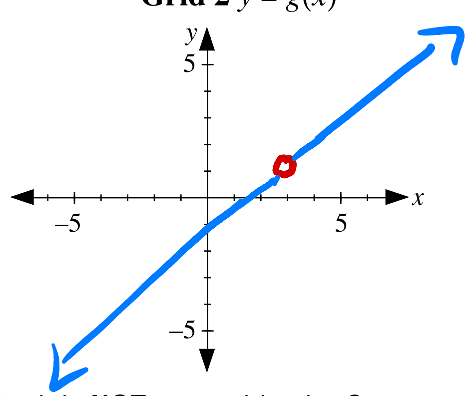
Graph $y = g(x)$ on Grid 2 using an open circle to represent the "hole", or discontinuity, of the graph. The "hole" in the graph is referred to as a **point of discontinuity**.

point of discontinuity @ $(3, 2)$

Grid 1 $y = f(x)$



Grid 2 $y = g(x)$



Point of Discontinuity

If the numerator and denominator of a rational function have a factor in common, then the graph of the rational function has a "hole" in it.

The point where the break in the graph occurs is called a **point of discontinuity** and the function is said to have **point discontinuity**.

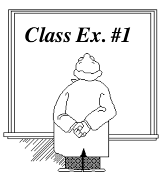
The point of discontinuity is represented on a graph by an **open circle**.

The coordinates of the point of discontinuity can be determined algebraically by using the following procedure:

1. Factor the rational expression.
2. Simplify the rational expression by cancelling the common factors.
3. Substitute the non-permissible value of x into the simplified form.

x in $\rightarrow$ y out

- When graphing a function with a point of discontinuity, the "hole" in the graph will usually NOT be seen unless the calculator is set to zoom decimal (`ZDecimal`), or a multiple of the zoom decimal window.
- If there are multiple points of discontinuity, the calculator may not show them all (especially in a multiple of the zoom decimal window).
  An algebraic verification is necessary to confirm the accuracy of the calculator graph.



Consider the rational function $f(x) = \dfrac{x^3 - 7x + 6}{x - 1}$.

a) Write the numerator of the function as a product of three binomial factors, and hence express $f(x)$ in simplest form.

$P(1) = 0$

$$\begin{array}{r|rrrr} 1 & 1 & 0 & -7 & 6 \\ & \downarrow & 1 & 1 & -6 \\ \hline & 1 & 1 & -6 & 0 \end{array}$$

$(x-1)(x^2+x-6)$

$(x-1)(x+3)(x-2)$

$$f(x) = \frac{\cancel{(x-1)}(x+3)(x-2)}{\cancel{x-1}}$$

$x \neq 1$

$$f(x) = (x+3)(x-2)$$

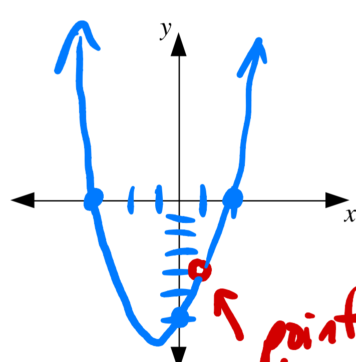
b) Determine the coordinates of the point of discontinuity of the graph of the function.

$x = 1 \rightarrow y = -4$

c) State the zeros of the function.

$x = -3, 2$

d) Use a graphing calculator to graph $y = f(x)$ and sketch the graph on the grid.



point of discontinuity @ $(1, -4)$

$\boxed{\#1\text{-}5}$