Analyzing Radical and Rational Functions Lesson #5: Graphs of Rational Functions

Review

If a rational function has a denominator which can be made equal to zero, then the graph of the rational functions will have a vertical asymptote, or a point of discontinuity corresponding to each zero of the denominator.

Asymptotes

Consider $f(x) = \dfrac{n(x)}{d(x)}$ **where $n(x)$ and $d(x)$ have no factors in common.**

Vertcial Asymptotes

Algebraically we can find the equations of vertical asymptotes of rational functions by finding the zeros of the denominator because the graph is undefined at those value(s). The equation(s) will be x = *the zero value*(*s*).

non-permissable or restricted

Horizontal Asymptotes

The graph of $f(x)$ has a horizontal asymptote under the following conditions:

- If the degree of $n(x)$ is less than the degree of $d(x)$, then the line $y = 0$ is a horizontal asymptote.
- If the degree of $n(x)$ is equal to the degree of $d(x)$, then the line $y = \dfrac{a}{b}$ is a horizontal asymptote, where a is the leading coefficient of $n(x)$ and b is the leading coefficient of $d(x)$.
- If the degree of $n(x)$ is greater than the degree of $d(x)$, then the graph has no horizontal asymptote.

Points of Discontinuity

Consider $f(x) = \dfrac{n(x)}{d(x)}$ **where $n(x)$ and $d(x)$ have a factor in common.**

The point where the break in the graph occurs is called a **point of discontinuity** and is represented on a graph by an **open circle**.

The coordinates of the point of discontinuity can be determined algebraically by using the following procedure:

1. Factor the rational expression.
2. Simplify the rational expression by cancelling the common factors.
3. Substitute the non-permissible value of x into the simplified form. → gives you the y-coordinate

Investigating Graphs of Rational Functions with No Discontinuities

If the denominator of a rational function has no zeros, there will be no discontinuities on the graph of the function.

Part 1: Consider the function $f(x) = \dfrac{1}{x^2 + 1}$.

Answer questions a) to f) before graphing the function.

a) Does the function have any non-permissible values?

none b/c $x^2+1 \neq 0$

b) State the domain of the function.

D: $x = \mathbb{R}$

c) The maximum value of the function will occur when the denominator has its lowest value. Determine the maximum value of the function.

lowest when $x=0$

$\therefore$ max $y=1$

d) Explain why the value of $f(x)$ can never be negative.

both numerator & denominator can only ever be +ive.

e) Does the function have any zeros?

none

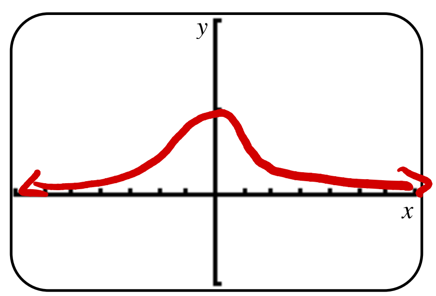
f) Use the answers above to suggest a possible range for the function.

$0 < y \leq 1$

g) Use a graphing calculator to graph $y = f(x) = \dfrac{1}{x^2 + 1}$

using window x:[−7, 7, 1], y:[−1, 2, 1].

Sketch the graph on the grid.



h) Does the graph of the function have

i) a vertical asymptote? **ii)** a point of discontinuity? **iii)** a horizontal asymptote?

i) no

ii) no

iii) yes $y=0$

Class Ex. #1

Consider the following functions.

$a(x) = \dfrac{x-4}{x+4}$ $\quad b(x) = \dfrac{x^2-4}{x^2+4}$ $\quad c(x) = \dfrac{x+4}{x^2-4}$ $\quad d(x) = \dfrac{x+2}{x^2-4}$ $\quad e(x) = \dfrac{x^2+4}{x+4}$

$x \neq -2$

$\dfrac{x+4}{(x+2)(x-2)}$ $\quad \dfrac{\cancel{x+2}}{\cancel{(x+2)}(x-2)} = \dfrac{1}{x-2}$

$f(x) = \dfrac{x^2+4}{x^2-4}$ $\quad g(x) = \dfrac{-x-4}{-x^2-4}$ $\quad h(x) = \dfrac{-x^2-4}{-x^2+4}$ $\quad i(x) = \dfrac{x^2-4}{x+2}$ $\quad j(x) = \dfrac{x^3-4x^2}{x^2-2x}$

$\dfrac{x^2+4}{(x+2)(x-2)}$ $\quad \dfrac{-(x+4)}{-(x^2+4)} = \dfrac{x+4}{x^2+4}$ $\quad \dfrac{-(x^2+4)}{-(x^2-4)} = \dfrac{x^2+4}{(x+2)(x-2)}$ $\quad x \neq -2 \quad \dfrac{\cancel{(x+2)}(x-2)}{\cancel{x+2}} = x-2$ $\quad \dfrac{\cancel{x^2}(x-4)}{\cancel{x}(x-2)} = \dfrac{x(x-4)}{x-2} \quad x \neq 0$

Without sketching the graph of the function, determine which functions have

a) no discontinuities b, g

b) no vertical asymptote b, g, i

c) no horizontal asymptote e, i, j

d) the x-axis as a horizontal asymptote c, d, g

e) a horizontal asymptote with equation $y = 1$ a, b, f, h

f) point(s) of discontinuity d, i, j

Class Ex. #2

Algebraically determine the equations of any asymptotes and the coordinates of any point(s) of discontinuity on the graph of the function $f(x) = \dfrac{(2x-1)(x-5)}{x^2-2x-15}$.

State the domain and range of f.

$f(x) = \dfrac{(2x-1)\cancel{(x-5)}}{\cancel{(x-5)}(x+3)}$ $\quad x \neq 5$

$f(x) = \dfrac{2x-1}{x+3}$

vert. asymp: $x = -3$

hor. asymp: $y = 2$

p.o.d: $(5, 9/8)$

D: $x = \mathbb{R}$ except $x \neq -3, 5$

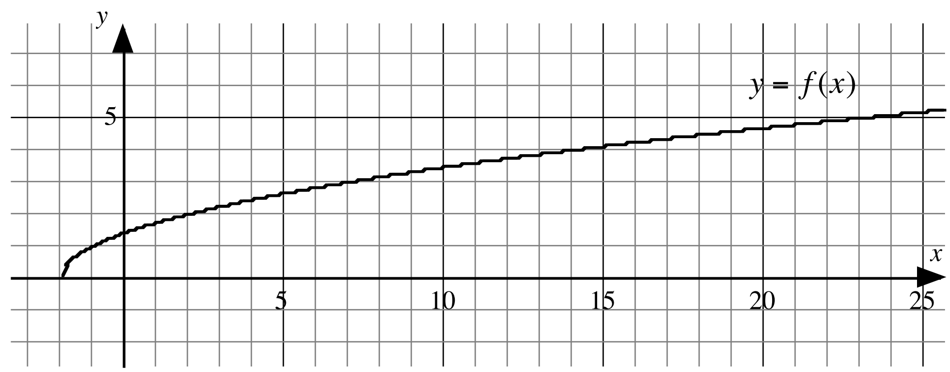
R: $y = \mathbb{R}$ except $y \neq 2, 9/8$

Complete Assignment Questions #1 - #5

$\boxed{\#1\text{-}3, 5}$

Class Ex. #2

A partial graph of $y = f(x)$, a radical function with an integral zero, is shown.



a) State the zero of the function.

b) State the root of the equation $f(x) = 0$.

c) Use the graph to determine, to the nearest whole number,

i) the solution of the equation $f(x) = 3$ **ii)** the root of the equation $f(x) - 4 = 0$

d) In each case, explain how to use the graph of $y = f(x)$ to determine the approximate solution to the given equation and state the solution to the nearest whole number.

i) $f(x + 6) = 0$

ii) $f(x - 3) = 5$

Class Ex. #3

Consider the functions $f(x) = 2x - 3$, $g(x) = x - 1$, and $h(x) = x + 5$.
Determine, graphically, the solutions of the following equations to the nearest hundredth.

a) $\left(\frac{f}{g}\right)(x) - 5 = 0$ **b)** $\cancel{\sqrt{f(x)} = (g \circ h)(x)}$ **c)** $\left(\frac{f}{g}\right)(x) = \left(\frac{g}{h}\right)(x)$

$$\frac{2x-3}{x-1} - 5 = 0$$

$$\frac{2x-3}{x-1} = 5$$

$$2x - 3 = 5x - 5$$

$$\frac{2}{3} = \frac{3x}{3}$$

$$\boxed{x = \frac{2}{3}}$$

$$\frac{2x-3}{x-1} = \frac{x-1}{x+5}$$

$$(2x-3)(x+5) = (x-1)(x-1)$$

$$2x^2 + 7x - 15 = x^2 - 2x + 1$$

$$x^2 + 9x - 16 = 0$$

quad. formula

$$\boxed{x = 1.52, -10.52}$$

Complete Assignment Questions #1 - #9

$$\frac{8}{p-8} = \frac{6}{p-3} - \frac{p-6}{p^2-11p+24}$$

$$\cancel{(p-8)}(p-3) \left[\frac{8}{\cancel{p-8}} = \frac{6}{\cancel{p-3}} - \frac{p-6}{\cancel{(p-8)(p-3)}} \right] \cancel{(p-8)(p-3)}$$

$$(p-8)\cancel{(p-3)}$$

$$(8p-24) = (6p-48) - (p-6)$$

$$8p-24 = 6p-48-p+6$$

$$\frac{3p}{3} = \frac{-18}{3}$$

$$\boxed{p=-6}$$

Practice → Complete

* KUTA worksheet