Exponential and Logarithmic Functions Lesson #4:
Logarithmic Functions

Exploring the Inverse of an Exponential Function

In this example we will consider the exponential function $y = 2^x$.

Part 1 **Exploring the Inverse of $y = 2^x$ Algebraically**

To find the inverse of a function algebraically, we must switch x and y and then solve for y.

a) Attempt to determine the inverse of $y = 2^x$ algebraically.

$x = 2^y$

b) What difficulty did you encounter?

we don't know how to solve.

At this stage we are unable to write the inverse of $y = 2^x$ in terms of y.

Part 2 **Exploring the Inverse of $y = 2^x$ Graphically**

To determine the inverse of $y = 2^x$ graphically, we switch the x and y-coordinates of each point on the graph to produce the graph of $x = 2^y$.

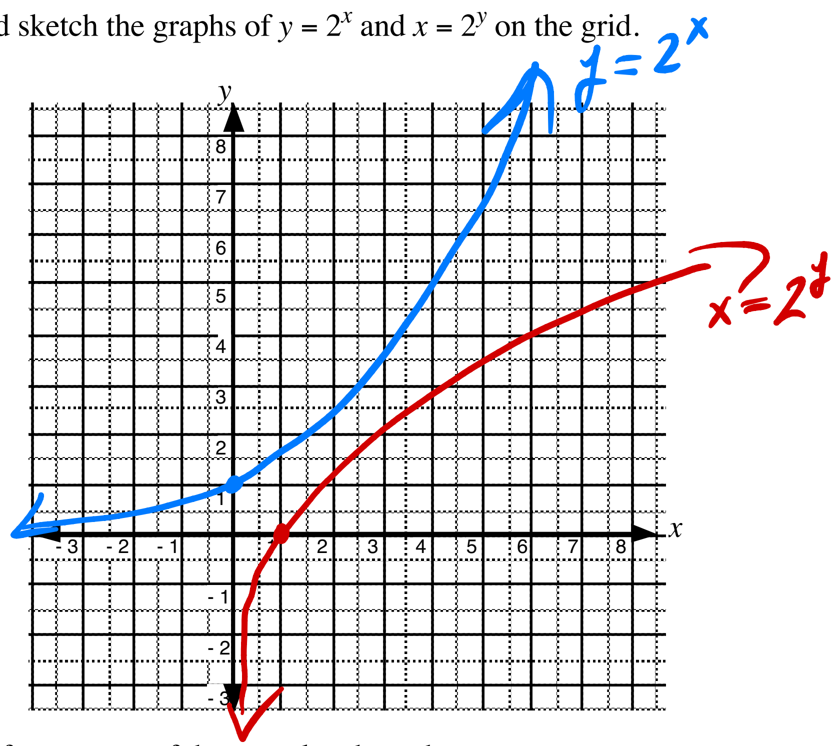
a) Complete the tables below and sketch the graphs of $y = 2^x$ and $x = 2^y$ on the grid.

Graph of $y = 2^x$

x	–3	–2	–1	0	1	2	3	4
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16

Graph of $x = 2^y$

x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16
y	–3	–2	–1	0	1	2	3	4



b) State the equation of the line of symmetry of the completed graphs.

$y = x$

$y = 2^x$ is the exponential function with base 2. The inverse of this function, $x = 2^y$, is also a function, but we are unable to write its equation in terms of y.

To do this, we introduce a new function, called the **logarithmic function**.

Logarithmic Function

A **logarithmic function** is the inverse of an exponential function.

The inverse of the exponential function with base 2, i.e. $y = 2^x$,
is the logarithmic function with base 2, written as $y = \log_2 x$.

Note that the graph of $y = \log_2 x$ is the same as the graph of $x = 2^y$.

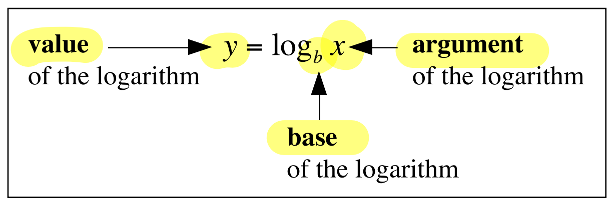
$$x = 2^y \quad \Leftrightarrow \quad y = \log_2 x$$

In general, we write $y = \log_b x$ rather than $x = b^y$ to express the inverse of $y = b^x$.

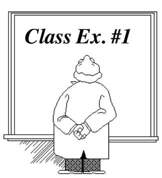
The **logarithmic function** with base b has the equation

$$y = \log_b x, \quad x > 0, \; x \in R, \; b > 0 \text{ and } b \neq 1$$

- The inside of the logarithm, in this case x, is called the **argument** of the logarithm.
- The argument can never be negative.



$y = \log_2 x$

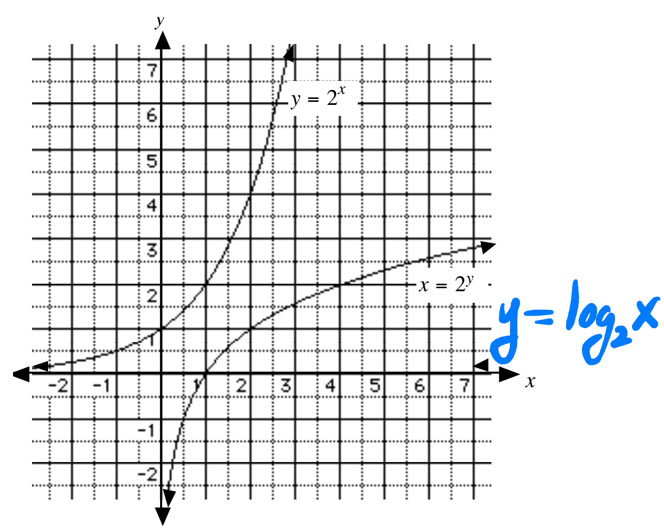


Class Ex. #1

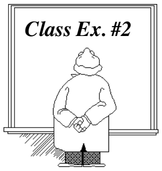
The graphs of $y = 2^x$ and $x = 2^y$ are shown.

a) Write the label "$y = \log_2 x$" beside the appropriate graph on the grid.

b) Complete the table below.



	Function	Domain	Range	x-intercept	y-intercept	Asymptote
exponential	$y = 2^x$	$x = \mathbb{R}$	$y > 0$	none	$(0,1)$	$y = 0$
logarithmic	$y = \log_2 x$	$x > 0$	$y = \mathbb{R}$	$(1,0)$	none	$x = 0$



The tables below show the coordinates of points on the graphs of $y = 2^x$ and $y = \log_2 x$.

Graph of $y = 2^x$

x	-3	-2	-1	0	1	2	3	4
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16

Graph of $y = \log_2 x$

x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16
y	-3	-2	-1	0	1	2	3	4

Notice that

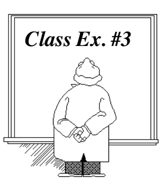
- the point $(3, 8)$ on the graph of $y = 2^x$ indicates that $8 = 2^3$
- the point $(8, 3)$ on the graph of $y = \log_2 x$ indicates that $3 = \log_2 8$

a) What statement can be made from the point $\left(-2, \frac{1}{4}\right)$ on the graph of $y = 2^x$?

b) What statement can be made from the point $(16, 4)$ on the graph of $y = \log_2 x$?

c) Complete the table below showing statements in exponential form and logarithmic form.

Logarithmic Form	**Exponential Form**	**Logarithmic Form**	**Exponential Form**
$\log_2 8 = 3$	$8 = 2^3$	$\log_2 \frac{1}{8} = -3$	$\frac{1}{8} = 2^{-3}$
$\log_2 4 = 2$	$4 = 2^2$	$\log_2 \left(\frac{1}{4}\right) = -2$	$\frac{1}{4} = 2^{-2}$
$\log_2 2 = 1$	$2 = 2^1$	$\log_2 \left(\frac{1}{2}\right) = -1$	$\frac{1}{2} = 2^{-1}$
$\log_2 1 = 0$	$1 = 2^0$		



a) Use patterns developed from Class Ex. #2 to write the exponential statement $10^3 = 1000$ in logarithmic form.

$\log 1000 = 3$

log base 10 is usually not written w/ a base.

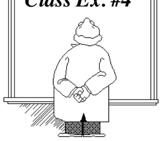
b) Use patterns developed from Class Ex. #2 to write the logarithmic statement $\log_5 625 = 4$ in exponential form.

$5^4 = 625$

Characteristics of the Graph of the Logarithmic Function $y = \log_b x$

- The x intercept is 1.
- There is no y-intercept.
- The y-axis is a vertical asymptote with equation $x = 0$.
- Domain = $\{x \mid x > 0, x \in R\}$.
- Range = $\{y \mid y \in R\}$.
- $y = \log_b x$ is equivalent to $x = b^y$, where $x > 0$ and $b > 0$, $b \neq 1$.
- b is the base of both the logarithmic function and the exponential function.

- **Since the logarithmic function $y = \log_b x$ is only defined for positive values of x, the logarithm of a negative number cannot be determined.**
- **The logarithmic equation $y = \log_b x$ can be expressed in exponential form as $x = b^y$.**
- **The exponential equation $y = b^x$ can be expressed in logarithmic form as $\log_b y = x$.**

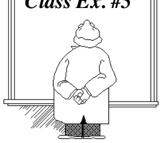


Class Ex. #4

State the inverse of the following functions. Answer in the form $y =$ ________ .

a) $y = \log_3 x$ **b)** $y = 8^x$

$x = \log_3 y$ $y = 3^x$ $x = 8^y$ $\log_8 x =$

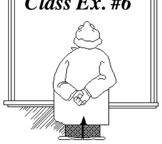


Class Ex. #5

Convert each of the following from logarithmic form to exponential form.

a) $\log_7 x = 4$ **b)** $\log_5 15 = y$ **c)** $m = \log_t B$ **d)** $\frac{5}{4} = \frac{4\log_b 6}{4}$

$x = 7^4$ $5^y = 15$ $t^m = B$ $b^{5/4} = 6$



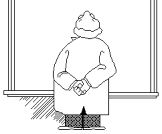
Class Ex. #6

Convert each of the following from exponential form to logarithmic form.

a) $4^3 = 64$ **b)** $2^{-3} = \dfrac{1}{8}$ **c)** $e^d = f$ **d)** $(2x + 4) = a^5$

$\log_4 64 = 3$ $\log_2 \frac{1}{8} = -3$ $\log_e f = d$ $\log_a (2x+4) = 5$

Class Ex. #7



Calculate the value of t if $\log_2 x = 3$ and $\log_2 t = x$.

$2^3 = x$

$x = 8$

$\log_2 t = 8$

$t = 2^8 = \boxed{256}$

Complete Assignment Questions #1 - #7

The Logarithmic Form of $y = ab^x$

We have seen how to change forms between the exponential form $y = b^x$ and the logarithmic form $\log_b y = x$.

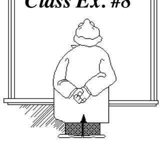
We now consider how to write the exponential form $y = ab^x$ in logarithmic form.

This can be done using the following procedure:

1. Write the exponential form $y = ab^x$ as $\frac{y}{a} = b^x$.
2. Change $\frac{y}{a} = b^x$ to logarithmic form.

The logarithmic form of $y = ab^x$ $\left(\text{or } \frac{y}{a} = b^x\right)$ is ____________________ .

Class Ex. #8



Change each of the following from exponential form to logarithmic form.

a) $y = 2(3^x)$

$\frac{y}{2} = 3^x$

$\log_3\left(\frac{y}{2}\right) = x$

b) $h = 7(4)^k$

$\log_4\left(\frac{h}{7}\right) = k$

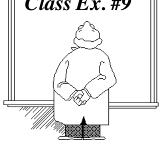
c) $t = r(s)^p$

$\log_s\left(\frac{t}{r}\right) = p$

d) $y = \frac{3}{2}(10)^x$

$\log_{10}\left(\frac{2}{3}y\right) = x$

Class Ex. #9



Change each of the following from logarithmic form to exponential form $y = ab^x$.

a) $\log_7\left(\frac{y}{3}\right) = x$

$7^x = \frac{y}{3}$

$y = 3(7)^x$

b) $\log_{10}\left(\frac{y}{4}\right) = x$

$10^x = \frac{y}{4}$

$y = 4(10)^x$

c) $\log_5(7y) = x$

$5^x = 7y$

$y = \frac{1}{7}(5)^x$

d) $\log_e\left(\frac{y}{5}\right) = x$

$e^x = \frac{y}{5}$

$y = 5(e)^x$

Complete Assignment Questions #8 - #15

#1-15 (a, c)